Engineering Notes

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Formulas for Spanwise Distribution of Lift on Aircraft Wings

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I. Introduction

THE motivation for the problem considered in this paper is taken from loads analysis for aircraft structures. The strength to be built into an aircraft structure depends on the type of maneuvers expected of the craft as set down in the specifications of the prospective buyer. From the consideration of the overall equilibrium of the aircraft, it is usually possible to determine the total lift on the vehicle. As a first approximation, all lift is assumed to be due to the wing, and the first design problem for the loads analyst is to find the spanwise load distribution for a given value of total lift.

Without loss of generality, the point of this paper is made by considering the (discrete) numerical solution for the integrand in the equation

$$\int_{[\text{wing span}]} l(y,\alpha)\mathrm{d}y = L_w(\alpha) \tag{1}$$

where L_w is the known total lift on the wing, y denotes distance along the span, and α is a parameter which will be referred to as the load source. For illustrative purposes, α can be taken to represent the aircraft angle of attack, but the interpretation placed on it can vary. For example, when the emphasis is on loads due to a control surface deflection, it may be taken to be the magnitude of that control surface (such as a flap) deflection.

Equation (1), as it stands, has many mathematical solutions. Physically, however, we know there has to be a unique solution. Removal of degeneracy from the problem is aided partly by the availability of more information and partly by the application of physical laws. For a wing of total span b, we can elaborate by writing

$$\int_{-b/2}^{b/2} C(y)C_l(y,\alpha)\mathrm{d}y = L_w(\alpha) \tag{2}$$

where $C(y)$ and $C_l(y,\alpha)$ are the spanwise distributions of chord lengths and sectional lift coefficients, respectively. (The similarity with the general linear Fredholm integral equation of the first kind is deceptive. Although Eq. (2) can be considered a special case of the Fredholm equation of the first kind, an interpretation of $C_l(y\alpha)$ as the kernel of the equation may be misleading from the physical point of view.) When the spanwise distributions of airfoil sections, chord lengths, and geometric twist are specified, the function $C_l(y,\alpha)$ and corresponding $L_w(\alpha)$ can be determined from aerodynamic theories for a range of values of α. In this case, the solution of Eq. (1) can be reduced to interpolations in α. However, the approach of tabulating $C_l(y,\alpha)$ and $L_w(\alpha)$ is not always feasible for at least two reasons: 1) computations are lengthy and expensive; and 2) the final design is more likely to be the offshoot of a series of iterations, during which the input needed in aerodynamic computations may not be completely known. Thus, the need to solve Eqs. (1) and (2) with insufficient information always exists. This paper presents a rational approach to this perennial design problem.

The approach is based on a solution of Eqs. (1) and (2), containing undetermined quantities which are then interpreted physically before the final determinate solution is chosen. The final (numerical) solution is given by a simple, explicit formula which can be programmed or evaluated with a pocket calculator. If $C_l(y,\alpha)$ and $L_w(\alpha)$ have been tabulated accurately from some aerodynamic theory for a set of values of α, the formula will actually interpolate and extrapolate the solutions to the same accuracy. If accurate aerodynamic data are not available, physical reasoning will usually provide a rational basis for a good approximate solution.

II. The Integrand as a Point on a Hyperplane

In seeking a numerical solution of the equation

$$\int_a^b f(y)\mathrm{d}y = J \tag{3}$$

for a given value of J, we will be content to characterize the integrand as a set of step functions defined over a finite number of subintervals of $[a,b]$. Thus, we are led to an equation of the form

$$f_1\Delta y_1 + f_2\Delta y_2 + \ldots + f_n\Delta y_n = J \tag{4}$$

where $[a,b]$ has been divided into n parts $\Delta y_1, \Delta y_2, \ldots, \Delta y_n$, and $f_1, f_2, \ldots, f_n$ are the unknown mean values of the function $f(y)$ in these intervals. In general, an equation such as Eq. (3), will lead to an equation of the form

$$A_1x_1 + A_2x_2 + \ldots + A_nx_n = C \tag{5}$$

where A_i, $i=1, n$ are known weights and x_i, $i=1, n$ are the corresponding unknowns in specified subintervals of $[a,b]$. Equation (5) represents a hyperplane in n-dimensional Euclidean space, and we present a characterization of points on the plane which can be adapted to solutions of physically meaningful problems, such as the one discussed in Sec. I.

Geometrically, a solution of Eq. (5) is a column vector in R^n. There are an infinity of solutions, since every point of the hyperplane is one such solution. A method of selecting a specific solution is required. Following a standard linear algebraic approach, let us construct a basis for the solutions of the corresponding homogeneous equation. (One may refer to Ref. 1 for this approach.) There are $(n-1)$ basis vectors:

$$v_1 = \begin{bmatrix} -A_2/A_1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -A_3/A_1 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ldots, v_{n-1} = \begin{bmatrix} -A_n/A_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{6}$$

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Given a particular solution of Eq. (5), for example,

$$X_p = \begin{bmatrix} C/A_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{7}$$

one can represent a point P on the plane by the expression

$$P = X_p + d_1 v_1 + d_2 v_2 + \ldots + d_{n-1} v_{n-1} \tag{8}$$

where $d_2, \ldots, d_{n-1}$ are constants. To set up a more concrete characterization, the points on the hyperplane are described as the projection of a given vector H along another vector B. In this way, a point on the plane can also be written in the form

$$H + tB \tag{9}$$

where t is a number that must be determined from the equation

$$H + tB = X_p + d_1 v_1 + \ldots + d_{n-1} v_{n-1} \tag{10}$$

There are n equations and n unknowns in Eq. (10). Given H and B, we need only to find t to determine the coordinates of a point on the plane. Let the components of H and B be, respectively,

$$H = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \tag{11}$$

Equation (10) then can be written in matrix form as:

$$\begin{bmatrix} -b_1 & -A_2/A_1 & -A_3/A_1 & \cdots & -A_n/A_1 \\ -b_2 & 1 & & 0 & \\ \cdot & & \ddots & & \\ \cdot & 0 & & \ddots & \\ \cdot & & & & \\ -b_n & & & & 1 \end{bmatrix} \begin{bmatrix} t \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 - C/A_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \tag{12}$$

The matrix on the left-hand side of Eq. (12) a simple form. It has nonzero elements along the first row and first column and 1's in the diagonal of the $(n-1) \times (n-1)$ submatrix obtained by deleting the first row and first column. All other elements are zero. By partitioning this matrix and solving explicitly for t, we get

$$t = -[N \cdot H - C]/N \cdot B \tag{13}$$

where the vector N is the normal to the plane given by

$$N = \begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} \tag{14}$$

The solutions to Eq. (5) can, therefore, be given in terms of H and B by the formula

$$P = H - \frac{[N \cdot H - C]}{N \cdot B} B = H + \frac{[N \cdot P - N \cdot H]}{N \cdot B} B \tag{15}$$

For a fixed H, all possible solutions can be covered by varying the vector B. The solution when $B = N$ may be considered optimum in the sense that it minimizes the Euclidean distance from H to the hyperplane. It is seen to correspond to a least-square solution.

Let up point out that from the mathematical point of view, any solution from Eq. (15) is as good as any other solution. In engineering applications, however, this may not be the case. Only certain solutions can be considered acceptable in any given situation. The vectors H and B play "natural" roles, making it possible to identify a smaller subset of acceptable solutions from a larger, infinite set. This is because the vectors can be given physical interpretations which allow for further deductions based on physical laws.

III. Physical Meaning of the Free Vectors

What are the vectors H and B, and how can we identify them in any given situation? Can we reasonably expect these vectors to be available? The answers to these questions are in the affirmative and we proceed to present them.

First, a physical significance can be attached to H. For example, suppose the right-hand side of Eq. (5) represents a scalar quantity like lift L. Then, since the n-dimensional space can be "filled" with hyperplanes parallel to the hyperplane of solutions of Eq. (5) such that each new plane corresponds to a distinct value of L, H can be interpreted as the (discrete) function for some distinct value of L, say, L^1. In this case, Eq. (15) can be written in the form

$$\Delta P = \frac{\Delta L}{N \cdot B} B \tag{16}$$

where

$$\Delta P = P - H \tag{17a}$$

$$\Delta L = L - L^I \tag{17b}$$

Thus, we solve for an increase in the functional distribution due to an increase ΔL in total lift.

In this context, the significance of the vector $\boldsymbol{B}$ can be deduced. Again, the lift on an aircraft wing will be used to motivate the interpretation; extensions to other situations are transparent. At a given Mach number, the total wing lift is a function of the detailed wing geometry as well as the aircraft angle of attack α. By fixing the wing geometry, the total lift varies only with α. This is the rationale for introducing α as a parameter in Eqs. (1) and (2). Obviously other parameters can be introduced as deemed necessary.

When a parameter is introduced, then an increase in L due to a change in this parameter can be obtained to a first approximation by differentiation. Changes in the distribution (density) can likewise be obtained by differentiating. For example, an increase in the spanwise lift distribution due to a change in angle of attack $\Delta\alpha$ will be $[\partial l(y,\alpha)/\partial\alpha]\ \Delta\alpha$, and by noting that

$$\Delta L = \frac{dL}{d\alpha}\Delta\alpha = \Delta\alpha \int_{-b/2}^{b/2} \frac{\partial l}{\partial \alpha}\ (y,\alpha)\, dy \tag{18}$$

we deduce that

$$\Delta\alpha = \Delta L \Big/ \int_{-b/2}^{b/2} \frac{\partial l(y,\alpha)}{\partial\alpha} dy \tag{19}$$

If we discretize the integral in Eq. (19) in exactly the same way as the integral of the equation we set out to solve, it will be found that the increase in spanwise distribution due to $\Delta\alpha$ in the interval i is:

$$\frac{\Delta L}{N \cdot F} f_i \tag{20}$$

where

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix} \tag{21}$$

and f_i is the average value of a function proportional to $\partial l/\partial\alpha$ in the ith interval. Comparing Eqs. (20) and (16), it is seen that the vector $\boldsymbol{B}$ can be thought of as representing a function proportional to $\partial l/\partial\alpha$. Observe that the actual value of $\partial l/\partial\alpha$ need not be known.

These interpretations of vectors $\boldsymbol{H}$ and $\boldsymbol{B}$ are useful for the following reason. Often these quantities can be computed or measured beforehand. In that case, Eqs. (15) and (16) become compact interpolatory and extrapolatory formulas. Savings in time and money result from the use of the simple, yet efficient, formula.

Another application of the interpretive solution is in obtaining and evaluating approximate solutions. Usually the initial solution $\boldsymbol{H}$ is known from the physical formulation. When not available, the function $\partial l/\partial\alpha$ can be guessed. It is likely to be a constant or slowly varying in order that upper and lower estimates can be readily made. From this approach, many imaginative and useful new approximations can be obtained. By casting existing approximations in the form of Eqs. (15) or (16), and comparing with known facts about $\partial l/\partial\alpha$, the degree of validity of such approximations can be easily established.

The geometrical ideas that lead to Eqs. (15) and (16) also reveal that perturbation solutions can be obtained without further information such as $\partial l/\partial\alpha$. Solutions close to $\boldsymbol{H}$ can be determined quite accurately by the simple choice

$$B = N \tag{22}$$

Such a choice is equivalent to a least-square solution.

Finally, it should be noted that other useful interpretations of the vector $\boldsymbol{B}$ are possible. One such interpretation is in terms of a prescribed shape or density function. If the shape of the dependent function is known or prescribed, then it can be shown by direct computation that $\boldsymbol{B}$ is the discrete representation of the shape function in the intervals of interest. Of course, when there is a parameter in the problem, the shape function and $\partial l/\partial\alpha$ must be proportional. This can sometimes be used as a crosscheck between different approximation techniques.

IV. Lift Distribution on a Wing at Subsonic Speeds

Returning to Eq. (2), we take a preliminary design problem in which estimates of the spanwise lift distribution on a wing or arbitrary planform are sought at different values of L_w. For simplicity, it is assumed that the distribution at $L_w = 0$ is known. (Note that this could be a nontrivial distribution.) The distribution of chord lengths $C(y)$ is also known, so that the basic unknown function is $C_l(y,\alpha)$. Although changes in L_w are known to be due to changes in aircraft angle of attack α, an explicit relationship between the two variables is assumed to be unknown at this point.

Following the results of Secs. II and III, it can be deduced that an estimate for the distribution $(\partial C_l/\partial\alpha)\ (y,\alpha)$ will be required. This is comforting because this function varies very slowly over the span. In fact, it is known to be constant for two-dimensional wings. In this particular case, we will assume that variations from the theoretically predicted two-dimensional result arise from two sources—finite-wing effects and wing-body interaction. Under these assumptions, $(\partial C_l/\partial\alpha)\ (y,\alpha)$ can be assumed to be a parabola defined in terms of values at the wing root, mid-semispan, and tip. In one such approximation, we took

$$a_t = \frac{\partial C_l}{\partial\alpha}\bigg|_{y=b/2} = \frac{2\pi}{1 + (2\pi S/\pi b^2)} \qquad (S = \text{wing area})$$

$$a_m = \frac{\partial C_l}{\partial\alpha}\bigg|_{y=b/4} = (1.7)\pi$$

$$a_r = \frac{\partial C_l}{\partial\alpha}\bigg|_{y=0} = (1.2)\pi \tag{23}$$

where α is in radians. The equation for the approximate function is:

$$\frac{\partial C_l}{\partial\alpha}(y) = a_r - \left[\frac{a_t + 3a_r}{2} - 2a_m\right]\left(\frac{|y|}{b/4}\right) + \left[\frac{a_t + a_r}{2} - a_m\right]\left(\frac{y}{b/4}\right)^2 \tag{24}$$

The preceding results were used only at subsonic speeds since the two-dimensional lift-curve slope had been assumed to be 2π. In one instance, at least, Eqs. (23) and (24) led to remarkably good results, considering the amount of work needed to achieve better results. This example has been given to show the ease with which useful approximations can be obtained. There is no claim that it is superior to traditional approximations, as will be pointed out later.

In Eq. (2), the unknown function is the section lift coefficient. Depending on the load source and problem formulation, different dependent functions arise naturally. In all cases, the dependent variable is directly related to lift per unit span. For this reason it may be regarded as some kind of generalized load density. The generalized load density may be a physically meaningful quantity whose variation can be determined experimentally or from a reliable empirical law. Consequently, the choice of the load density could be an important factor in obtaining acceptable solutions.

In considering some traditional approximations, the generalized load density will be simply lift per unit span. For this density, experience shows that the elliptic distribution, planform distribution, and the Schrenk approximation are sometimes good approximate solutions. Following the second interpretation of B in Sec. III, these approximations can be easily cast in the form of Eqs. (15) or (16). For the elliptic distribution, B represents the function

$$f(y) = \sqrt{1-(2y/b)^2} \qquad (25)$$

and for the planform distribution, it represents $C(y)$. The Schrenk approximation is the average of the elliptic and planform distributions. These facts were verified and the three approximations were compared with the first [Eq. (24)] and a more elaborate computer result. All approximations did so remarkably well that establishing a preference between them was not easy. The ranking went like this: Schrenk, Eq. (24), elliptic and planform (tied).

Application of the least-square solution, Eq. (22), to spanwise lift distribution showed that the adequacy of this general solution depends a great deal on the choice of the generalized load density. Very good results were obtained when $C_l(y,\alpha)$ was used. The use of lift per unit span itself led to very poor or just marginal results. This is a clear indication that mathematical solutions chosen at random can be dangerous.

Given the guidelines stated in this paper, many useful applications can be derived from the solutions just presented. More applications can be found in Ref. 2.

V. Conclusions

There are two parts to the results of this study. The first gives a compact formula specifying a function in terms of its definite integral in a given domain. In this formula, the inherent mathematical indeterminacy of the problem under consideration was isolated in two free terms. The second part of the result is the demonstration that pathological mathematical solutions can be avoided by the successful linkage of the free terms to physically meaningful parameters characteristic of a given problem. Thus, the study provides a rational basis for engineering approximations of detailed function distributions from their definite integrals which may be known from measurements. Since numerical computations with the formula are very simple, an experienced engineer has the opportunity to obtain highly accurate design estimates at great savings in time and money.

Although only the spanwise lift distribution on aircraft wings has been considered here, it should be clear that many other applications are possible. Identifying new applications, especially formulas with empirical characterizations of the free terms in the formula will be valuable contributions.

References

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